JOURNAL OF COMPUTATIONAL PHYSICS 10, 65–84 (1972)

Information Loss and Compensation in Linear Interpolation

RALPH SHAPIRO

Air Force Cambridge Research Laboratories
L. G. Hanscom Field, Bedford, Massachusetts 01730

Received June 23, 1971

A general operator of arbitrary order is developed for the purpose of minimizing the deleterious truncation and aliasing effects introduced by interpolation. The effects of interpolation are discussed for both uniform and mixed grid systems and the properties of the restoration operator are demonstrated by means of a series of computations on the sea-level pressure distribution around latitude circles. For geophysical applications, a low ordered operator is shown to be effective in restoring physically significant information which is lost during interpolation. It is inferred from the properties of the restoration operator that it may find useful application in avoiding computational instability in cases where interpolation is involved.

1. INTRODUCTION

Interpolation occupies a prominent role in numerical weather prediction. It enters in the process of specifying the initial data over some regular network and is actively involved in the numerical integration of the finite difference equations. It is an essential process when used to obtain boundary information in overlapping, multiple-map representations of the Earth's surface or in fine-mesh limited area, nesting problems.

All interpolation procedures selectively alter the amplitude spectrum and frequently the phase spectrum as well. In some applications the alterations are of no consequence and may actually be beneficial. For example, the simplest linear interpolation acts as a low pass filter. Consequently, when used as part of an objective analysis procedure to obtain initial data over some regular array, it serves to damp the high-frequency, noise-bearing component of the data and thus may actually help preserve the physical integrity of the numerical solutions. On the other hand, when used to obtain boundary information essential in the numerical integration, the characteristics of the interpolation process may contribute to or initiate the development of numerical instability. It is common experience that small-scale instability tends to develop near the interpolation boundary of overlapping gridsystems, whether the overlap is between different map projections [3] or between differing resolutions [4]. Though it is not certain that the instability is produced by the interpolation, it is certainly adversely affected by it; since in both cases the interpolation operates on one field which is then used in conjunction with an uninterpolated field. Thus, since the interpolation alters the spectrum, spurious gradients, especially in the short wavelengths, are introduced between the interpolated and noninterpolated fields. These spurious gradients will have a destabilizing effect on the computations and should be minimized. The purpose of this paper is to discuss a method of efficiently minimizing the deleterious effects of interpolation. The method consists essentially in finding the response or transfer function of the interpolation operator and modifying it by a series of simple, symmetrical linear filters which approximate the inverse of the response function.

2. One-Dimensional Interpolation

Given some function $Z(x)$, with data at uniformly spaced gridpoints i, $i+1,\ldots$, we may express the interpolated value at any point $i+r$ between i and $i+1$ as,

$$\hat{Z}_{i+r} = rZ_{i+1} + (1-r)\,Z_i\,, \qquad 0 \leqslant r \leqslant 1. \tag{1}$$

If we express $Z(x)$ in terms of Fourier components it is easily seen that the response function of the interpolation defined by (1) is given by

$$\rho_0(k) = [1 - 4r(1-r)\sin^2(k\Delta x/2)]^{1/2}, \tag{2}$$

where $k = 2\pi/L$ is the wave number corresponding to a wave of length L and ρ_0 is the ratio of the amplitude of $\hat{Z}_{i+r}$ to that of Z_i.

The maximum value of $4r(1-r)$ is 1 and occurs for $r = \frac{1}{2}$. Therefore, $\rho_0(k)$ is real and $0 \leqslant |\rho_0(k)| \leqslant 1$ for all k.

$\rho_0(k)$ shows how the amplitude spectrum of $Z(x)$ is altered by two-point linear interpolation. However, in as much as the spectrum of $Z(x)$ is only defined over some domain, $\rho_0(k)$ must be interpreted as if the operation defined by (1) had been performed for each point $i+r$ throughout the same domain. Furthermore, the expression (2) has validity only if r is constant in this domain.

It is apparent that for r near $\frac{1}{2}$, the interpolation defined by (1) can produce considerable damping of $Z(x)$, particularly for the higher wavenumbers. It is possible to restore much of the damping produced by (1) by applying an operator which approximates the inverse of (2). In order to maintain symmetry it is desirable to express this operator in terms of symmetrical three-point operators of the form

$$\bar{Z}_i = Z_i + \frac{S}{2}\,(Z_{i-1} + Z_{i+1} - 2Z_i), \tag{3}$$

where S is a smoothing element to be determined.

The response function corresponding to (3) is

$$1 - 2S\sin^2(k\Delta x/2). \tag{4}$$

If we apply the operator (3) to (1) and use the approach outlined in a previous paper [5], we find that in order to produce the maximum restoration of damped amplitude without amplification of any wave component, $S = -t/4$, where $t = 4r(1 - r)$. Thus, the response function for the first level of restoration of two-point linear interpolation is

$$\rho_1(a) = \rho_0(a)(1 + (t/2)\sin^2 a), \tag{5}$$

where $a = k\Delta x/2$. The combined operator may be treated as a three-point operator acting on a two-point operator or as a single four-point operator whose coefficients are cubic functions of r.

It is of interest to point out that the operator whose response function is given by $\rho_1(a)$ is identical with the 4-point Lagrange interpolation formula for the case where $r = \frac{1}{2}$ as well as for the trivial cases of $r = 0$ or 1. For all other values of r, the 4-point operator corresponding to the first level of restoration is superior to the 4-point Lagrange formula in the sense that $|\rho_1(a)| > |L_1(a)|$ for all $r \neq \frac{1}{2}, 0, 1$ for all a, where $L_1(a)$ is the response function corresponding to the 4-point Lagrange formula. The proof is given in Appendix A.

The next higher level of restoration is found by the same procedure to involve a complex conjugate pair of smoothing elements and is represented by

$$\rho_2(a) = \rho_1(a)(1 - 2S_2\sin^2 a)(1 + 2S_2\sin^2 a), \tag{6}$$

where $S_2{}^2 = -3t^2/32$. The combined operator represented by $\rho_2(a)$ may be considered either as a single eight-point operator whose coefficients are polynomials of seventh degree in r or as a two-point operator in conjunction with three separate three-point operators.

It is possible to continue the process of restoration indefinitely. The next higher order of restoration involves three additional three-point operators, and the h-th level of restoration involves h three-point operators, additional to those of the $h - 1$ level. Thus the operator at the h-th level of restoration consists of a two-point operator and $0 + 1 + 2 + 3 + \cdots + h = \sum_{j=1}^{h} j$ three-point operators. Each of the appropriate $\sum j$ smoothing elements can be determined level by level using the procedure outlined above. However, a simpler and more direct approach is outlined in Appendix B.

Although the process of restoration can be carried out indefinitely, it is necessary to truncate the process. The optimum truncation level is dictated by the application, but it is possible to draw some conclusions from an examination of the response function for the first few levels of restoration. Figures 1 and 2 show

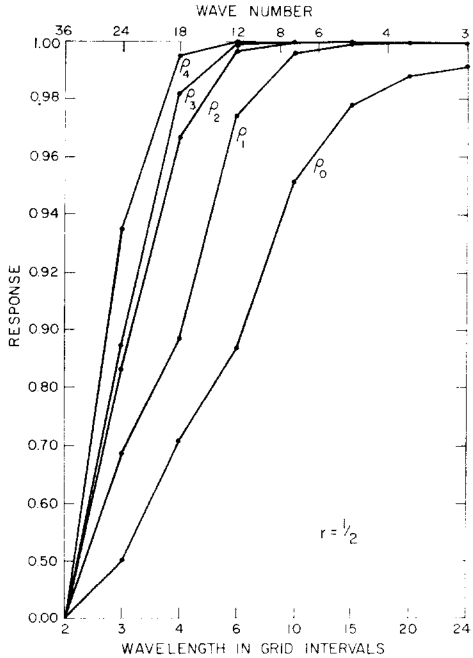

FIG. 1. Values of the response function $\rho_h(L)$ for levels of restoration $h = 0, 1, 2, 3$ and 4 for $r = \frac{1}{2}$.

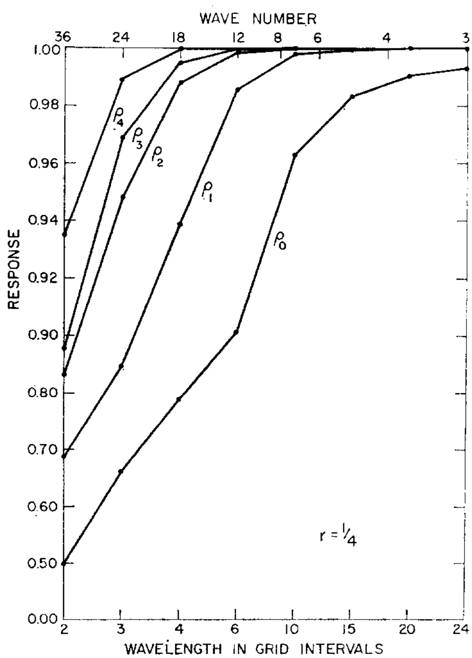

FIG. 2. Same as Fig. 1 for $r = \frac{1}{4}$.

some values of the response function for levels of restoration $h = 0, 1, 2, 3$ and 4 for $r = \frac{1}{2}$ and $\frac{1}{4}$. It is seen that ρ_2 represents a considerable improvement over ρ_1 which in turn represents a considerable improvement over ρ_0. However, ρ_3 and ρ_4 show little improvement over ρ_2 for wavelengths greater than 4 grid intervals. In geophysical applications, where energy density spectra generally decrease with increasing frequency, there would appear to be little reason for extending the restoration beyond the second level. This conclusion is substantiated in a subsequent section where interpolation and restoration are performed on the sea-level pressure distribution.

3. Heterogeneous Gridspacing

The expressions for the response functions for interpolation, with or without restoration, have been developed for the case where r is constant in the domain, or in other words, where we interpolate from a system of uniform gridspacing to another with the same spacing. These conditions are not satisfied in most applications, where frequently, one of the gridsystems is irregular. However, even if both are uniform but with different gridspacings, r is not a constant, but a function of space. In this case it is not a simple matter to find an analytical expression for the response function, since such an expression must allow for an unspecified number of gridpoints in the interpolation domain and incorporate the effects of the variation of r from point to point in this domain. We can, of course, perform a Fourier analysis on the original data and on the interpolated data and determine the set of numbers $\hat{A}(k)/A(k)$ which constitute the response function. However, it is not always possible, because of the shape of the domain in which we must operate to determine $A(k)$, let alone $\hat{A}(k)$. Furthermore, if we were to determine $A(k)$, the rationale for performing the interpolation would be obviated since we could directly determine the required value at any point from the Fourier phase and amplitude information. Therefore, let us assume that there exists for the case of variable r a series of expressions that are related in some unspecified way to $\hat{A}(k)/A(k)$ and which have the property that they reduce to the response function (2) when r is constant. If we choose to represent $\hat{A}(k)/A(k)$ by an expression which is simple and similar in form to (2) then we can find restoration operators comparable to (5) and (6) which are likely to have similar properties. We have arbitrarily chosen the expression

$$\rho_0^*(a, r) = [1 - 4r_\alpha(1 - r_\alpha)\sin^2 a]^{1/2} \tag{7}$$

to represent the response function $\hat{A}(k)/A(k)$. In (7), r_α is the value of r at the gridpoint α. Since the choice of (7) was arbitrary we shall not try to justify it on any but pragmatic grounds. Furthermore, we recognize that it is only one of a number of possible choices and not necessarily the best of these. Nevertheless, we shall show that the choice of (7) leads to reasonable restoration operators and what is more important to reasonable results.

The expression (7) cannot be called a response function since it depends upon r as well as wavenumber, however, it may be usefully interpreted as follows. At any gridpoint α, the effect of two-point interpolation on the amplitude spectrum would be given by (7) if we were to interpolate throughout the domain at a series of gridpoints with a constant value of r equal to r_α. At any other gridpoint $\alpha + p$, we interpret (7) as if we had interpolated at a series of points throughout the domain with the value of r appropriate to the point $\alpha + p$. Since each gridpoint in the α system has a specific value of r which may be different from every other value of r, the effect of carrying out two-point interpolation throughout the domain is to damp the same wave component by different amounts at each gridpoint. From a spectral point of view this process is tantamount to the introduction of wave components in the α domain that did not exist in the original i domain as well as altering the phases of the original wave components. Since the magnitude of the fictitious new components introduced by the interpolation, as well as the magnitude of the phase shifts, depends upon the difference in the amount of damping from one α point to the next, these undesirable aspects of the interpolation can be minimized by the restoration process since in the limit all the original amplitudes (not completely removed by interpolation) are restored to their original values regardless of position. For any practical application, it can be shown by examination of the magnitude of the change of response function with $r(|\partial\rho_h^*/\partial r|)$ that the magnitude of the fictitious component for any level of restoration $h > 0$, is less than that for $h = 0$ for all components longer than the 3 grid-interval wave. Furthermore, when $h > 1$, the magnitude of the fictitious component can be shown to be less than that for $h = 0$ for all components, longer than the 2 grid-interval wave. The exception for the 2 grid-interval wave arises because for this component, $\rho_h^* = 0$ for $r = \frac{1}{2}$ for all levels of restoration h, and inasmuch as $0 \leqslant \rho_{h-1} \leqslant \rho_h^* \leqslant 1$, and $\rho_h^* = 1$ for $r = 0$ and 1 for all h, it follows that $|\partial\rho_h^*/\partial r| > |\partial\rho_0^*/\partial r|$ in the vicinity of $r = \frac{1}{2}$ for the 2 grid-interval wave component. The increase in the magnitude of the fictitious component for the 2 grid-interval wave when $h > 0$ should cause no difficulty since if the grid distance is properly chosen the 2 grid-interval wave should contain little or no physically significant information and should be removed or damped by some suitable filter. Thus, it appears likely that in problems such as the fine-mesh nesting problem, where 2-point interpolation is used to obtain the necessary boundary information and where of necessity r is a function of position, the fictitious component is an important element in boundary instability. With the aim of reducing the magnitude of these fictitious components, we examine the expressions for the quasi-response functions $\rho_h^*(a, r)$ for h equal to 1 and 2 when r is a function of position.

$$\rho_1^*(a, r) = \rho_0^*(a, r)(1 - 2S_1 \sin^2 Ma), \tag{8}$$

$$\rho_2^*(a, r) = \rho_1^*(a, r)(1 - 4S_2^2 \sin^4 Ma), \tag{9}$$

where now

$$S_1 = -t_\alpha/4M^2, \qquad S_2^2 = -3t_\alpha^2/32M^4, \qquad M = nm,$$

n is the smallest integer for which $M \geqslant 1$ and m is the ratio of the α gridspacing to that of the original i gridspacing.

4. Multidimensional Operations

In two dimensions we define a linear interpolation at the point $i + r, j + p$ in the x, y plane as:

$$\hat{Z}_{i+r,j+p} = prZ_{i+1\,j+1} + p(1-r)\,Z_{ij+1} + (1-p)\,rZ_{i+1j} + (1-p)(1-r)\,Z_{ij}\,, \tag{10}$$

where r is defined (as in the one-dimensional case) as the fraction of the distance of the point $i + r$ between the points i and $i + 1$ and p is similarly defined with respect to the points j and $j + 1$. If $Z(x, y)$ is represented in two-dimensional waveform as

$$Z_{ij} = C + A \cos k(X_i - \varphi) \cos w(y_i - \theta),$$

where C is a constant, A is the amplitude of the two-dimensional wave with wave number k and phase φ in the x-direction and wave number w and phase θ in the y-direction, the quasi-response function corresponding to (10) at the point $(\alpha, \beta) = (i + r, j + p)$ is

$$\rho_0^*(k, w; \alpha, \beta) = [(1 - t_\alpha \sin^2 a)(1 - t_\beta \sin^2 b)]^{1/2}, \tag{11}$$

where

$$t_\alpha = 4r_\alpha(1 - r_\alpha),$$
$$t_\beta = 4p_\beta(1 - p_\beta),$$
$$a = k\,\Delta x/2,$$
$$b = w\,\Delta y/2,$$

and the quasi-response functions corresponding to the restoration operations for $h = 1$ and 2 are

$$\rho_1^*(k, w; \alpha, \beta) = \rho_0^*(k, w; \alpha, \beta)(1 - 2S_{1x} \sin^2 M_x a)(1 - 2S_{1y} \sin^2 M_y b), \tag{12}$$

$$\rho_2^*(k, w; \alpha, \beta) = \rho_1^*(k, w; \alpha, \beta)(1 - 4S_{2x}^2 \sin^4 M_x a)(1 - 4S_{2y}^2 \sin^4 M_y b), \tag{13}$$

where

$$S_{1x} = -t_\alpha/4M_x^2; \qquad S_{1y} = -t_\beta/4M_y^2;$$
$$S_{2x}^2 = -3t_\alpha^2/32M_x^4; \qquad S_{2y}^2 = -3t_\beta^2/32M_y^2;$$
$$M_x = nm_x; \qquad M_y = vm_y;$$

n and v are the smallest integers for which nm_x and vm_y are, respectively, equal to or greater than 1. m_x and m_y are, respectively, the ratios of the distances of the point α, β from i to the x grid distance, Δx, and from j to the y-grid distance, Δy. The extension of these expressions to any number of dimensions is obvious.

5. Applications

To determine the effectiveness of the restoration process in practice and in particular where $m \neq 1$, a number of simple one-dimensional experiments were carried out. They all involve the sea-level pressure distribution at 5 degree longitude intersections at latitudes 35, 45 and 55N on two separate dates, Dec. 22, 1969 and Jan. 9, 1970.

In the first experiment, the original data P_i, ($i = 1, 2, \ldots, 72$) are transformed to $\hat{P}_\alpha$ using (1). Five different gridspacings are used for the α gridsystem. These five systems each have their first point in common; namely, the point corresponding to $i = 1 + \Delta x/3$. The spacing of each of the 5 systems is then determined by dividing the latitude circle into 70, 60, 50, 40 and 36 equal grid intervals respectively.

TABLE I

Mean square differences $D_0 = 1/72 \sum_{i=1}^{72} (P_i - \hat{P}_i)^2$ mb^2 and $D_2 = 1/72 \sum_{i=1}^{72} (P_i - \tilde{\hat{P}}_i)^2$ mb^2 for three latitudes on two dates for five grid densities N

	$N = 70$		$N = 60$		$N = 50$		$N = 40$		$N = 36$	
	D_0	D_2	D_0	D_2	D_0	D_2	D_0	D_2	D_0	D_2
Dec.										
35N	0.39	0.09	0.51	0.28	0.58	0.21	1.05	0.58	1.45	0.81
45N	0.72	0.11	1.03	0.27	1.26	0.33	2.39	0.90	3.42	1.19
55N	0.42	0.06	0.65	0.23	1.04	0.38	1.48	0.51	1.94	0.87
Jan.										
35N	1.02	0.27	1.29	0.72	1.64	0.89	1.92	1.31	4.20	2.75
45N	1.07	0.40	1.05	0.33	2.31	1.36	3.36	2.19	4.38	3.05
55N	0.70	0.10	0.95	0.31	0.94	0.32	2.47	0.75	2.90	0.82
AVE	0.72	0.17	0.91	0.36	1.30	0.58	2.11	1.04	3.05	1.58

Each of the 30 sets of $\hat{P}_\alpha$ are then transformed to $\hat{P}_i$, again making use of (1). Thus, each set of $\hat{P}_i$ consists of doubly interpolated pressure values at the original 72 i-gridpoints. The first interpolation is from a 72-gridpoint system with grid-spacing Δx to a system occupying the same physical domain with a smaller number of gridpoints. Therefore, in the first interpolation $m > 1$. However, in the second interpolation, proceeding from the α to the i gridsystem, $m < 1$.

In addition to the 30 sets of $\hat{P}_i$ data, 30 sets of $\bar{\hat{P}}_i$ data were also evaluated by separate transformations of the $\hat{P}_\alpha$ data. That is, using the stencil corresponding

TABLE II

Variance of original pressure E(in mb^2) and the relative variance as a fraction of E as a function of grid density N at each stage in the interpolation and restoration process

$$A_0 = \hat{E}/E;\ A_2 = \bar{\hat{E}}/E;\ B_0 = \hat{\hat{E}}/E;\ B_2 = \bar{\hat{\hat{E}}}/E$$

	Dec.			Jan.			
	35N	45N	55N	35N	45N	55N	Ave.
E	56.0	239.5	360.6	136.7	254.6	288.7	
$N = 70$							
A_0	0.97	0.98	0.99	0.97	0.98	0.98	0.98
A_2	1.00	1.00	1.00	1.00	1.00	1.00	1.00
B_0	0.94	0.97	0.98	0.95	0.97	0.96	0.96
B_2	0.99	1.00	1.00	0.99	1.00	1.00	1.00
$N = 60$							
A_0	0.98	0.98	0.99	0.98	0.99	0.98	0.98
A_2	1.00	1.00	1.00	1.00	1.00	1.00	1.00
B_0	0.95	0.96	0.98	0.96	0.98	0.95	0.96
B_2	0.99	0.99	1.00	0.99	1.00	1.00	1.00
$N = 50$							
A_0	0.96	0.98	0.99	0.97	0.98	0.99	0.98
A_2	1.00	1.00	1.00	1.00	1.00	1.00	1.00
B_0	0.91	0.95	0.98	0.93	0.96	0.96	0.95
B_2	0.98	1.00	1.00	0.98	0.99	1.00	0.99
$N = 40$							
A_0	0.97	0.99	0.99	0.99	0.99	0.97	0.98
A_2	0.99	1.00	1.00	1.01	1.00	0.99	1.00
B_0	0.91	0.94	0.96	0.94	0.95	0.92	0.94
B_2	0.98	0.99	1.00	1.00	1.00	0.98	0.99
$N = 36$							
A_0	0.97	0.98	0.99	0.94	0.97	0.98	0.97
A_2	1.00	1.00	1.00	0.96	0.99	1.00	0.99
B_0	0.88	0.92	0.96	0.88	0.92	0.90	0.91
B_2	0.99	0.99	1.00	0.96	0.98	1.00	0.99

to the operator, Eq. (9), the $\hat{P}_\alpha$ data are restored to $\bar{\hat{P}}_\alpha$. The latter data are then interpolated onto the i gridsystem, using (1) and these last values are then restored again using the restoring operator corresponding to (9). In the first use of the restoring operator to obtain $\bar{\hat{P}}_\alpha$, $m > 1$ and consequently none of the $\hat{P}_\alpha$ values are omitted in the process of restoration. However, in the restoration process to obtain $\bar{\hat{P}}_i$, $\frac{1}{2} \leqslant m < 1$, and therefore the value of M used in (9) is obtained with $n = 2$. In the discussion which follows, unless specified otherwise, it should be understood that interpolation with restoration refers to the second level of restoration, equivalent to an 8-point operator in one dimension.

Inasmuch as P_i, $\hat{P}_i$ and $\bar{\hat{P}}_i$ all involve data at the same gridpoints, it is possible to compare the data and determine both the effects of interpolation and restoration from one gridsystem to another and back to the original system. Table I which contains the means of the squared differences $1/72 \sum (P_i - \hat{P}_i)^2$ and $1/72 \sum (P_i - \bar{\hat{P}}_i)^2$, shows such a comparison. It is apparent that the restoration process goes a long way toward recovering information lost in the process of interpolation and that the restoration process is most effective when N, the number of gridpoints in the α domain is greatest.

Table II shows the variance of the original P_i data as well as the fraction of the original variance at each step in the interpolation and restoration process. With only one interpolation (A_0) around 2 to 3 % of the variance is lost, but the restoration process (A_2) effectively recovers almost all of the original variance. With the second interpolation (B_0) an additional 2 to 6 % of the variance is lost, but the doubly interpolated and restored values (B_2) still contain almost all of the original variance.

Figure 3 demonstrates the spectral properties of interpolation, with and without restoration, using the December data for $45N$. The upper part of the figure shows the variance spectrum of P_i on a log-ordinate scale. The remaining curves are relative variance spectra showing the variance of $\hat{P}_i$ (represented by crosses) and the variance of $\bar{\hat{P}}_i$ (represented by open circles) as a fraction of the original variance in wave numbers 1–18. Because of the very small values of variance of P_i in the high wave numbers, the relative spectral estimates for these wave numbers tend to be erratic and have been omitted from the figure.

It is apparent that without restoration, considerable loss of variance is experienced in the baroclinically important wavenumbers (4–8). However, with restoration, the variance in each of the wave numbers (1–8) is effectively restored to its original value. There is some damping, even with restoration, at the higher wave numbers, but in every case the restored spectral values are closer to the original than the unrestored values.

The original data P_i is a specific sampling from some unknown continuous function $P(X)$. $P(X)$ undoubtedly contains information that is not resolved by P_i and which is fictitiously represented as longer wavelength components in the P_i

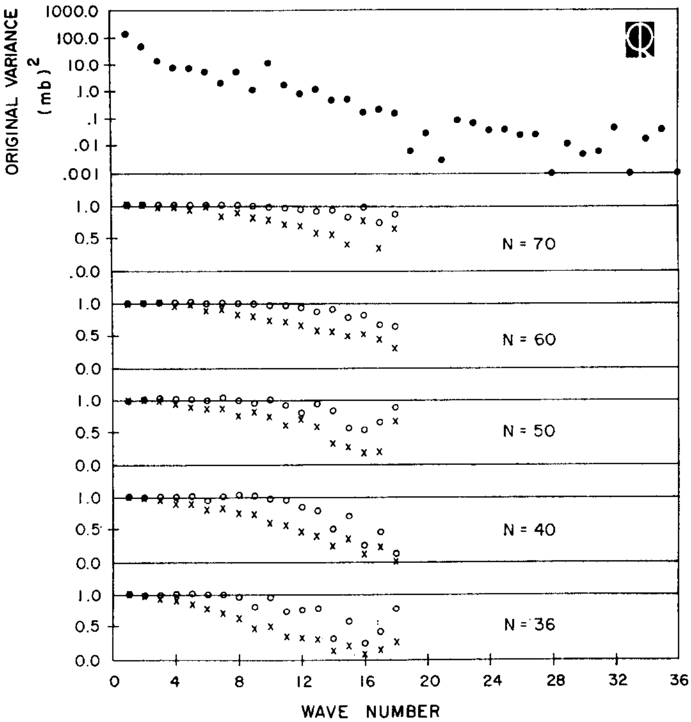

FIG. 3. The uppermost curve is the variance spectrum of the sea-level pressure (P_i) around 45N on 22 Dec 1969 evaluated from observations at every 5 deg of longitude. The remaining curves show the relative variance (as a fraction of the spectral estimate of P_i) of doubly interpolated pressures ($\hat{P}_i$, crosses) and doubly interpolated and restored pressures ($\bar{\hat{P}}_i$, open circles) for various interpolation gridpoint densities N.

spectrum. Furthermore, if some set of P_α were known, where P_α differs from P_i only by some constant shift in position, the variance spectrum of $P(X)$ obtained from P_α would undoubtedly be somewhat different from that obtained with P_i because the amount and distribution of aliased variance would undoubtedly be different. As a result of such aliasing, some of the relative spectral values in Fig. 3 exceed unity both in the unrestored as well as restored spectra. Furthermore, the fictitious components referred to above which are introduced by 2-point interpolation will also contribute to minor alterations in the spectra. However, inasmuch as the restoration process reduces the magnitude of the aliased variance contributed from both of these sources, it produces spectra in which not only the amplitudes but also the phases are more nearly the same as those of the original values. For example, for the case of $N = 70$, the mean phase shift for wave numbers 1 through 6 in terms of 360 degrees per wavelength is less than 0.01 degrees for $\bar{P}_i$, but 0.40 degrees for $\hat{P}_i$; for wave numbers 7 through 12, the comparable numbers are 0.38 and 3.45 degrees; and for wave numbers 13 through 18, the phase shifts average 1.97 and 6.98 degrees respectively. Thus, for those waves containing most of the variance, the phase shifts with restoration are negligible, while those without restoration, although small, are not negligible. For the higher wave numbers, which contribute little to the total variance, the phase shifts may not be negligible even with restoration, but they are substantially reduced by the restoration. Thus it appears, both from the direct comparisons between P_i and $\hat{P}_i$ and $\bar{P}_i$ as well as from the spectral distributions of these parameters, that the restoration operation succeeds in undoing much of the damage introduced by interpolation. Furthermore, from the spectral results, it appears that with second-level restoration, significant damping of amplitude and change of phase is limited to the higher wave components. These components not only contain little variance in the original data, but the variance that is present is undoubtedly highly contaminated by noise. Thus, the damping of these components may actually be beneficial.

The results presented up to this point were all obtained with the initial interpolation domain (α) containing fewer gridpoints than the original gridpoint system (i). In these results, N, the number density of the interpolation domain, takes on the values 70, 60, 50, 40 and 36, while the original data are distributed over 72 gridpoints. Thus the interpolation grid-lengths are all greater than the 5 degree grid-length of the original data. In the next set of results, the same original data are interpolated onto a series of domains wherein the gridlengths are smaller than 5 degrees, and which contain 74, 86, 104, 130 and 144 gridpoints, respectively.

TABLE III

Mean square differences $D_0 = 1/72 \sum_{i=1}^{72} (P_i - \hat{P}_i)^2\ mb^2$ and $D_2 = 1/72 \sum_{i=1}^{72} (P_i - \bar{P}_i)^2\ mb^2$ for grid densities N

	$N = 74$		$N = 86$		$N = 104$		$N = 130$		$N = 144$	
	D_0	D_2	D_0	D_2	D_0	D_2	D_0	D_2	D_0	D_2
Dec.										
35*N*	0.21	0.03	0.19	0.03	0.18	0.02	0.08	0.00	0.12	0.00
45*N*	0.54	0.09	0.53	0.05	0.29	0.02	0.23	0.00	0.25	0.00
55*N*	0.56	0.12	0.36	0.05	0.20	0.02	0.16	0.00	0.17	0.00
Jan.										
35*N*	0.78	0.38	0.49	0.10	0.40	0.06	0.21	0.01	0.31	0.00
45*N*	0.87	0.22	0.82	0.18	0.40	0.05	0.36	0.01	0.35	0.00
55*N*	0.69	0.15	0.40	0.05	0.28	0.03	0.17	0.00	0.23	0.00
Ave.	0.61	0.16	0.46	0.08	0.29	0.03	0.20	0.00	0.24	0.00

The ratios of these new gridlengths to the basic data grid-interval were chosen to be approximately equal to the reciprocals of the comparable ratios in the first experiment. The first point of each α set was placed at the same point as in the first experiment; namely, one-third the distance between $i = 1$ and $i = 2$. Table III which is comparable to Table I summarizes the results of the second experiment.

The benefits of restoration are even more apparent in these new results. The mean-square differences of both the interpolated and the interpolated and restored values generally decrease with increasing N. However, the decrease of this "error" is greater with the restored values, resulting in essentially no "error" for the higher values of N.

An analysis of the total variance of the doubly interpolated and restored pressures ($\tilde{\hat{P}}_i$), shows that it is essentially identical to that of the original data (within 0.2% on the average). But the variance of the unrestored pressures differs from the original variance by 2.5% on the average.

The spectral properties of $\hat{P}_i$ and $\tilde{\hat{P}}_i$ are shown in Fig. 4 for N greater than 72. The uppermost spectrum which is the same as that in Fig. 3 is repeated for ease

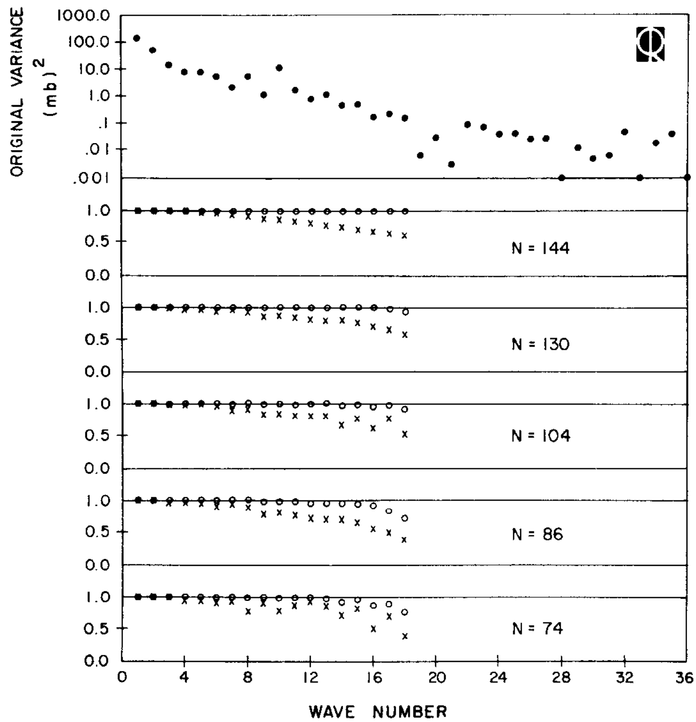

FIG. 4. Same as Fig. 3 for N greater than 72.

in reading. It is apparent from the relative spectra in Fig. 4 that there is little aliasing in either the restored or unrestored spectral estimates, especially for the higher values of N. Consequently, the spectra of the restored values are virtually identical to the original spectra. The lack of aliasing is a direct consequence of the smaller gridspacing and increased resolution of the interpolation domains. This feature of the results is clearly shown in Table IV which compares the magnitudes of the departures of amplitude and phase for doubly interpolated data, both with and without restoration, as a function of wavelength and number density of the interpolation domain.

TABLE IV

Average of the magnitude of the departures of amplitude (in mb) $[\Delta A]$ and phase (in deg) $[\Delta\varphi]$ from the original values after interpolation, both with and without restoration for $N = 74$ and 130[a]

	$N = 74$		$N = 130$	
	Wavenumbers			
	1–18	19–36	1–18	19–36
$[\Delta A]$	0.14	0.13	0.09	0.08
$[\Delta A^*]$	0.02	0.09	0.01	0.02
$[\Delta\varphi]$	2.8	31.5	1.1	12.9
$[\Delta\varphi^*]$	0.5	20.7	0.3	1.4
$[\Delta A^*]/[\Delta A]$	0.12	0.69	0.07	0.23
$[\Delta\varphi^*]/[\Delta\varphi]$	0.18	0.66	0.24	0.11

[a] The asterisk indicates restored values.

It is apparent from Table IV that there is a substantial decrease in the magnitude of both the phase and amplitude "error" with $N = 130$ as compared with $N = 74$, regardless of whether the restoration operation has been applied. If we examine the relative "error," shown as a ratio of the magnitudes of the restored to unrestored values, we see that the benefit of restoration is very pronounced for $N = 74$ for the low wavenumbers. The relative amplitude and phase error are, respectively, 0.12 and 0.18. However, for the high wavenumbers, substantial "error" remains even after restoration; the relative errors being 0.69 and 0.66, respectively. With $N = 130$, the relative "error" in the low wavenumbers is almost the same as that with $N = 74$. Although there is some decrease in the relative amplitude error (0.12 to 0.07), there is an increase in the relative phase error (0.18 to 0.24). On the other hand, there is a large improvement in the relative error of the high wave numbers with $N = 130$ for both amplitude (0.69 to 0.23) and phase (0.66 to 0.11).

6. Results with Higher Orders of Restoration

It was tentatively concluded on the basis of Figs. 1 and 2 that if the original data contained only minor contributions from the shorter wave components, orders of restoration higher than $h = 2$ would not be useful. To test this hypothesis, results comparable to those shown in Tables I and III, where the restoration was carried to order $h = 2$, were extended to orders $h = 3$ and 4. The average results for both dates and all three latitudes are shown in Fig. 5, along with the comparable results for $h = 0$ and 2 taken from Tables I and III as well as the results for $h = 1$. It is readily apparent that there is little or no reduction of the "error" for $h = 3$ or 4 as compared with $h = 2$. However, the results with $h = 2$ represent a substantial improvement over those with $h = 1$ which in turn represents an even larger improvement over those with $h = 0$. These results are, therefore, completely in agreement with the conclusion derived from Figs. 1 and 2.

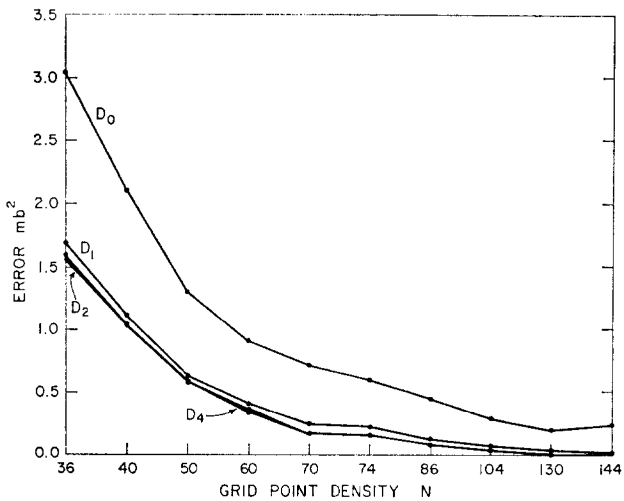

Fig. 5. Mean-square differences (in mb²)(D_h) for $h = 0, 1, 2, 4$. Values are the averages for all three latitudes for both dates. D_3 which is not shown falls between D_2 and D_4.

8. Conclusions

A general operator of arbitrary order h has been developed which is "ideal" in the sense of restoring a field subjected to linear interpolation as close as desired to its original spectral distribution without amplifying any wave component. For geophysical applications a low ordered operator with $h = 1$ or 2 has been shown to be very effective in restoring the physically significant components. The greatest virtue of the restoration operator, however, may lie in its damping of the fictitious components introduced by interpolation in domains with differing gridspacings. To the extent that such fictitious components contribute to computational instability in finite-difference solutions, the restoration process may be found to be an effective palliative.

Appendix A

Using the same notation as in (1), the stencil for the 4-point Lagrange interpolation operator is

$$\hat{Z}_{i+r} = \frac{-r(r-1)(r-2)}{6} Z_{i-1} + \frac{(r^2-1)(r-2)}{2} Z_i \quad \frac{-r(r+1)(r-2)}{2} Z_{i+1} + \frac{r(r^2-1)}{6} Z_{i+2}\,. \tag{1A}$$

The square of the response function corresponding to (1A) is

$$L_1^2(a) = 1 - \tfrac{12}{9} r(1-r^2)(2-r)\sin^4 a - \tfrac{16}{9} r^2(1+r)(1-r)^2(2-r)\sin^6 a, \tag{2A}$$

whereas $\rho_1^2(a)$, the square of the response function corresponding to the first level of restoration of two-point linear interpolation, is

$$\rho_1^2(a) = 1 - 12r^2(1-r)^2 \sin^4 a - 16r^3(1-r)^3 \sin^6 a. \tag{3A}$$

Since $0 \leqslant r \leqslant 1$, the coefficients of the $(-\sin^4 a)$ and $(-\sin^6 a)$ terms are positive for both $L_1^2(a)$ and $\rho_1^2(a)$. The ratios of the coefficients of $\rho_1^2(a)$ divided by the corresponding coefficients of $L_1^2(a)$ are equal and are given by

$$F(r) = 9r(1-r)/(1+r)(2-r). \tag{4A}$$

The maximum value of $F(r) = 1$ and occurs with $r = \frac{1}{2}$. Thus, for $r = \frac{1}{2}$, $\rho_1(a)$ and $L_1(a)$ are identical, but for all other values of r, $F < 1$. Thus, since the coefficients of the $(-\sin^4 a)$ and $(-\sin^6 a)$ terms are smaller with $\rho_1^2(a)$ than with $L_1^2(a)$ for all r except $r = \frac{1}{2}$, $|\rho_1(a)| \geqslant |L_1(a)|$.

Appendix B

The response function for two-point linear interpolation is from (2)

$$\rho_0(a) = (1 - t \sin^2 a)^{1/2}. \tag{1B}$$

We wish to find a series of symmetrical three-point operators of the form of (3) whose response functions are of the form (4). We call this combined operator $\prod_{j,h=1}^{h} O_{j,h}$, where h indicates the level of restoration and j is an index. If we wish to restore the amplitude of each wave component damped by (1) without amplifying any component, then the response function $\prod_{j,h=1}^{h} R_{j,h}$, corresponding to $\prod O_{jh}$ should approximate the inverse of $\rho_0(a)$. That is,

$$\prod_{j,h=1}^{h} R_{j,h}(a) \cong (1 - t \sin^2 a)^{-1/2}. \tag{2B}$$

The right side of (2B) is approximated by the series

$$1 + \frac{t}{2} \sin^2 a + \frac{1 \cdot 3}{2 \cdot 4} t^2 \sin^4 a + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} t^3 \sin^6 a + \cdots$$
$$+ \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2h-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2h)} t^h \sin^{2h} a$$

for all values of t and a except where $t \sin^2 a = 1$. Thus, $\rho_h(a) = \rho_0(a) \prod_{j,h=1}^{h} R_{j,h}(a)$ approaches 1 as h becomes large except for the two grid-interval wave for the case of $t = 1$ $(r = \frac{1}{2})$, for which $\rho_h(a)$ remains zero.

The series above can be expressed as a product of terms,

$$\left(1 + \frac{t}{2} \sin^2 a\right)\left(1 + \frac{3t^2}{8} \sin^4 a\right)\left(1 + \frac{t^3}{8} \sin^6 a\right)$$
$$\times \left(1 + \frac{27t^4}{128} \sin^8 a\right) \times \cdots \times (1 + C_h t^h \sin^{2h} a)$$

minus terms of higher order than $t^h \sin^{2h} a$. The coefficients $C_1, C_2, \ldots, C_h$ are determined so that $\rho_h(a) = 1$ minus terms of higher order than $t^h \sin^{2h} a$.

It is apparent that the above product of terms is $\prod_{h=1}^{h} R_h(a)$, where

$$R_h(a) = 1 + C_h t^h \sin^{2h} a = \prod_{j=1}^{h} (1 - 2S_{jh} \sin^2 a). \tag{3B}$$

The system (3B) consists of h equations for the h unknown values of S_{jh}. These equations are

$$\sum_{j=1}^{h} S_j = 0,$$

$$\sum_{\substack{j=1,2,\dots,h-1,\\ k=2,3,\dots,h,}} S_j S_k = 0, \qquad j \neq k,$$

$$\vdots$$

$$\sum_{\substack{j=1,2,\dots,h+1-p,\\ k=2,3,\dots,h+2-p,\\ m=3,4,\dots,h+3-p,\\ \vdots \\ p=p,p+1,\dots,h,}} S_j S_k S_m \cdots S_p = 0, \qquad j \neq k \neq m \neq \cdots \neq p,$$

$$\vdots$$

$$S_1 S_2 S_3 \cdots S_h = (-t/2)^h \, C_h \,.$$

It is apparent from (3B), however, that the h values of S_j are the h roots of $(-t/2)^h\, C_h$ and are given by (see, for example, [2])

$$S_j = \left| \frac{t}{2} C_h^{1/h} \right| \exp\left[i \left(\frac{\pi}{h} + \frac{2j\pi}{h} \right) \right], \qquad j = 1, 2, \dots, h. \tag{4B}$$

We see from (4B) that when h is odd, one value of S is real and the remaining values form complex conjugate pairs. When h is even, all values of S form complex conjugate pairs. Furthermore, if there is a real root it lies on the negative real axis of the complex plane and the complex roots have directions such that for any h the complex plane is divided into equal segments by all h roots.

It is a relatively simple matter to find all S_{jh} for any level of restoration h. We have from (1B), (2B) and (3B),

$$\rho_h(a) = \rho_{h-1}(a)\, R_h(a). \tag{5B}$$

If we know $\rho_{h-1}(a)$, C_h is easily determined from the series expansion of (1B) and the series approximation to (2B) using the requirement that C_h is such as to eliminate the $t^h \sin^{2h} a$ term from the expansion of $\rho_h(a)$. S_j can then be determined directly from (4B) for any level of restoration, h. Once all S_j are known, the restoration process can be carried out as $\sum_{j=1}^{h} j$ successive applications of the 3-point operator (3) to the field of $\hat{z}_{i+r}$ or as a single operator acting on z_i and involving $2(1 + \sum j)$ points.

The first derivative of the function representing two-point linear interpolation corresponds to a forward difference and thus depends upon whether the approach to the point i is from the right or left. It is easily seen, however, that the first derivative of the function representing any level of restoration is continuous at the endpoints. From (1) and (3) we have

$$(\overset{-1}{\hat{Z}}_{i+r})' = \overset{\prime}{\hat{Z}}_{i+r} + \frac{S'_{1,1}}{2}(\hat{Z}_{i-1+r} + \hat{Z}_{i+1+r} - 2\hat{Z}_{i+r})$$

$$+ \frac{S_{1,1}}{2}(\overset{\prime}{\hat{Z}}_{i-1+r} + \overset{\prime}{\hat{Z}}_{i+1+r} - 2\overset{\prime}{\hat{Z}}_{i+r}), \quad (6B)$$

where $\overset{\prime}{\hat{Z}}_{i+r} = Z_{i+1} - Z_i$ and $S'_{1,1} = 2r - 1$.

As r approaches zero and i is approached from the right

$$(\overset{-1}{\hat{Z}}_{i+r})' \to (\overset{-1}{\hat{Z}}_i)' = \tfrac{1}{2}(Z_{i+1} - Z_{i-1}) \quad (7B)$$

Furthermore, as $r \to 1$, the function $(\overset{-1}{\hat{Z}}_{i-1+r})' \to (\overset{-1}{\hat{Z}}_i)'$ which is also found by substitution into (6B) to equal $\frac{1}{2}(Z_{i+1} - Z_{i-1})$. Similarly, for the second level of restoration we have

$$(\overset{-2}{\hat{Z}}_{i+r})' = (\overset{-2*}{\hat{Z}}_{i+r})' + \frac{S'_{2,2}}{2}(\overset{-2*}{\hat{Z}}_{i-1+r} + \overset{-2*}{\hat{Z}}_{i+1+r} - 2\overset{-2*}{\hat{Z}}_{i+r})$$

$$+ \frac{S_{2,2}}{2}(\overset{-2*}{\hat{Z}}_{i-1+r} + \overset{-2*}{\hat{Z}}_{i+1+r} - 2\overset{-2*}{\hat{Z}}_{i+r})', \quad (8B)$$

where

$$\overset{-2*}{\hat{Z}}_{i+r} = \overset{-1}{\hat{Z}}_{i+r} + \frac{S_{1,2}}{2}(\overset{-1}{\hat{Z}}_{i-1+r} + \overset{-1}{\hat{Z}}_{i+1+r} - 2\overset{-1}{\hat{Z}}_{i+r}). \quad (9B)$$

From the definition of $S_{1,2}$ and $S_{2,2}$ we find by substitution of (9B) into (8B) that when the point i is approached either from the right or left, $(\overset{-2}{\hat{Z}}_i)' = (\overset{-1}{\hat{Z}}_i)'$ and, therefore, the first derivative of the function representing the second level of restoration is also continuous at the endpoints. Similarly, it can be shown that the first derivative of any higher level of restoration $(\overset{-h}{\hat{Z}}_{i+r})'$, can be reduced to a form equal to $(\overset{-1}{\hat{Z}}_{i+r})'$ plus terms involving products of powers of t. Since all such terms involving products of powers of t reduce to zero when r equals zero or one, it can be shown that $(\overset{-h}{\hat{Z}}_{i+r})'$ is continuous at the endpoints for any level h. Similarly, it can also be shown that $(\overset{-h}{\hat{Z}}_{i+r})''$ is not continuous at the endpoints for any level h.

Acknowledgments

I am indebted to my colleague Dr. C. H. Yang for many fruitful discussions, and for his efforts in rescuing me from the morass of mixed gridspacings.

REFERENCES

1. M. ABRAMOWITZ AND I. A. STEGUN, (Eds), "Handbook of Mathematical Functions." Table 26.11, 991, NBS Appl. Math. Ser. 55, U. S. Dept. Commerce, Washington, DC, 1964.
2. G. H. HARDY, "A course of pure mathematics," 10 ed., University Press, Cambridge, England, 1958.
3. N. A. PHILLIPS, *Mon. Wea. Rev.* **87** (1959), 333.
4. M. A. SHAPIRO AND J. J. O'BRIEN, *J. Appl. Meteorol.* **9** (1970), 345.
5. R. SHAPIRO, *Rev. Geophys. and Space Phys.* **8** (1970), 359.
6. R. SHAPIRO, Compensation of interpolation by linear filter, Phys. Sci. Res. Pap. No. 443, AFCRL-71-0058, Air Force Systems Command, L. G. Hanscom Field, Bedford, MA, 1971.